Phys 410
Spring 2013
Lecture #36 Summary
22 April, 2013

We considered another coupled oscillator problem – the double pendulum. We wrote down the Lagrangian, which turned out to be quite complicated. It leads to nonlinear equations of motion – as is well known for the single pendulum. To avoid this problem (which we will deal with later), we made a "small oscillations" approximation for the double pendulum. In this approximation we take ϕ_1, ϕ_2, $\dot{\phi}_1$, and $\dot{\phi}_2$ to be small, and only keep terms up to second order in these quantities. We then did a Taylor series expansion for the kinetic energy and potential energy to arrive at an approximate Lagrangian of the form: $\mathcal{L} = \frac{1}{2}(m_1 + m_2)\left(L_1\dot{\phi}_1\right)^2 + m_2 L_1 L_2 \dot{\phi}_1\dot{\phi}_2 + \frac{1}{2}m_2\left(L_2\dot{\phi}_2\right)^2 - \frac{(m_1+m_2)gL_1\phi_1^2}{2} - \frac{m_2 g L_2 \phi_2^2}{2}$. Both the kinetic energy and the potential energy are homogeneous quadratic functions.

We then used Lagrange's equations to find the equations of motion for the two generalized coordinates ϕ_1, ϕ_2, with the following results:

ϕ_1-equation: $-(m_1 + m_2)gL_1\phi_1 = (m_1 + m_2)L_1^2\ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2$

ϕ_2-equation: $-m_2 g L_2 \phi_2 = m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2$

These two equations can be summarized in matrix form as $\overline{\overline{M}}\,\ddot{\vec{\phi}} = -\overline{\overline{K}}\,\vec{\phi}$, with $\vec{\phi} = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$, $\overline{\overline{M}} = \begin{pmatrix}(m_1 + m_2)L_1^2 & m_2 L_1 L_2\\ m_2 L_1 L_2 & m_2 L_2^2\end{pmatrix}$ and $\overline{\overline{K}} = \begin{pmatrix}(m_1 + m_2)gL_1 & 0\\ 0 & m_2 g L_2\end{pmatrix}$. The "mass matrix" is now made up of rotational inertia terms, while the "spring constant matrix" is made up of restoring torque terms. We again use the complex ansatz for the solution vector: $\vec{\phi}(t) = Re\left[\vec{C}e^{i\omega t}\right]$, where $\vec{C} = \begin{pmatrix}C_1\\ C_2\end{pmatrix}$, and C_1 and C_2 are complex constants. Putting this into the matrix equation yields $\left(\overline{\overline{K}} - \omega^2\overline{\overline{M}}\right)\vec{C} = 0$. To get a non-trivial solution for $\vec{C}$, we demand that $det\left(\overline{\overline{K}} - \omega^2\overline{\overline{M}}\right) = 0$. This yields a quadratic equation for ω^2, with two solutions.

We then considered the special case of a double pendulum with equal masses (m) and equal lengths (L), and introduce the natural frequency ($\omega_0^2 \equiv g/L$). The determinant yields two normal mode frequency solutions: $\omega_1 = \omega_0\sqrt{2 - \sqrt{2}}$, and $\omega_2 = \omega_0\sqrt{2 + \sqrt{2}}$. The corresponding normal modes are the analogs of the "sloshing" and "beating" modes. The first is of the form $\vec{\phi} = A_1\begin{pmatrix}1\\ \sqrt{2}\end{pmatrix}\cos(\omega_1 t - \delta_1)$, while the second is $\vec{\phi} = A_2\begin{pmatrix}1\\ -\sqrt{2}\end{pmatrix}\cos(\omega_2 t - \delta_2)$. In the first normal mode the two pendula swing together in

phase (the sloshing mode), with the lower pendulum swinging with greater amplitude. In the other mode the two pendula swing 180^o out of phase (a type of beating mode).

We then went on to consider the most general coupled oscillator problem – N particles coupled to each other by means of springs or any other types of forces that produce a stable equilibrium configuration. This system has n generalized coordinates, where in general $n \neq N$. The generalized coordinates are written as $\vec{q} = (q_1, q_2, \ldots q_n)$. We assume that only conservative forces act between the particles, hence (as known from previous studies) the potential energy is a function only of the coordinates: $U = U(\vec{q})$. The kinetic energy is that of all of the particles in the system: $T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha^2$. The "raw" coordinates $\vec{r}_\alpha$can be written in terms of the generalized coordinates as $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \ldots q_n)$, where it is assumed that no explicit time-dependence is required to write down this transformation. The kinetic energy can be written as $T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, \dot{q}_i \dot{q}_j$, where the matrix $\bar{\bar{A}}$ is defined as $A_{ij} \equiv \sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i}\frac{\partial \vec{r}_\alpha}{\partial q_j}$. Note that the double pendulum kinetic energy (see the Lagrangian above) has a kinetic energy of this form, including a $\dot{q}_1\dot{q}_2$ term. Note that the matrix $\bar{\bar{A}}$ is a function of the generalized coordinates as well: $\bar{\bar{A}} = \bar{\bar{A}}(\vec{q})$. We now have the full Lagrangian of this generalized coupled oscillator problem $\mathcal{L} = T\left(\vec{q}, \dot{\vec{q}}\right) - U(\vec{q})$.